14. Rapoport, I. M., On a class of singular integral equations. Dokl. Akad. Nauk SSSR, Vol. 59, №8, 1948.
15. Whittaker, E. T. and Watson, G. N., Modern Analysis, Vol. 1, Moscow, Fizmatgiz, 1963 (Russian translation).

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VARIATIONAL METHODS OF CONSTRUCTING MODELS OF SHELLS (*)

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The purpose herein is to derive the relationships of the theory of elastic shells from the variational equation of the mechanics of continuous media in the general case of physically and geometrically nonlinear models. The examination of this question is interesting in connection with the fact that all the hypotheses acquire the most compact and explicit formulation in the variational approach, and a logical basis appears for the comparison and estimation of the various models proposed in the theory of shells. Moreover, the shell models yield an interesting illustration of models of continuous media in which there are firstly higher derivatives, and secondly, internal degrees of freedom originate, as will be seen later. The appearance of the internal degrees of freedom requires the establishment of additional equations, in addition to the ordinary equations of mechanics, in order to determine new parameters, and to raise the order of the differential equations – additional boundary conditions and conditions on discontinuities. These relationships have been obtained by using methods developed for arbitrary models of continuous media with internal degrees of freedom and with higher derivatives in [1, 2]. Let us note that the extension of the theory to inelastic shells is associated only with complicating the functional δW^* in (1.1) and adding new degrees of freedom due to plastic deformations, viscous deformations, etc. Only the general part of the theory is contained herein. Specific shell models will be examined separately.

1. Variational equation in the theory of elastic bodies. The fundamental relationships of the theory of elastic bodies can be obtained from the variational equation [1 – 3]

$$\delta \iint_{tV} \Lambda \, d\tau \, dt + \delta W^* + \delta W = 0 \tag{1.1}$$

where the Lagrangean Λ and the functional δW^* are the given quantities, and δW is an integral of a linear combination of the variations in the displacements over the bound-

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ary of the four-dimensional domain $V \times t$ and is found from (1.1). If the difference between the kinetic and integral energies

$$\Lambda = \rho_0 \left(\frac{w^{\cdot 2}}{2} - U \right) \tag{1.2}$$

is taken as the Lagrangean, then for models of elastic bodies δW^* is the sum of the work of the external mass forces on the possible displacements and the heat influx

$$\delta W^* = \iint\limits_{tV} \rho_0 (T \delta S + F_i \delta w^i)\, d\tau\, dt \tag{1.3}$$

while δW is the sum of the work of the external surface forces on the possible displacements and the work of the momenta at the initial and final times

$$\delta W = \int\limits_t \int\limits_{\partial V} p_i \delta w^i\, d\sigma\, dt - \left[\int\limits_V I_i \delta w^i\, d\tau \right]_{t_1}^{t_2} \tag{1.4}$$

Here and henceforth, V is an arbitrary associated volume, ∂V is its boundary, ρ_0 is the density of the medium in the initial state, w^i, I_i, p^i, F^i are the components of the displacements, momenta, and external surface and mass forces in the Cartesian reference system of the observer x^i, S is the entropy, and T is the temperature.

For models of elastic bodies the internal energy U is a function of the entropy S, some given parameters of the medium K_B, and the strain tensor components ε_{ij}

$$U = U(\varepsilon_{ij}, S, K^B), \qquad \varepsilon_{ij} = {}^1/_2 (g_{\wedge ij} - g_{(0)ij}) \tag{1.5}$$

$$g_{\wedge ij} = g_{kl} \frac{\partial r^k}{\partial \zeta^i} \frac{\partial r^l}{\partial \zeta^j}, \quad g_{(0)ij} = g_{kl} \frac{\partial r_0^k}{\partial \zeta^i} \frac{\partial r_0^l}{\partial \zeta^j}, \quad w^i(\zeta^j, t) = r^i(\zeta^j, t) - r_0^i(\zeta^j)$$

Here ζ^i are the Lagrangean coordinates of the particles, $x^i = r^i(\zeta^j, t)$ is the law of particle motion, $x^i = r_0^i(\zeta^j)$ is the initial position of the particles. Henceforth, for simplicity adiabatic processes will be considered. The entropy S is considered specified, and therefore goes over into a number of parameters K^B. Since $\delta S = 0$, the functional δW^* becomes

$$\delta W^* = \iint\limits_{tV} \rho_0 F_i \delta w^i\, d\tau\, dt \tag{1.6}$$

Assignment of the boundary conditions reduces to assigning δW when a domain corresponding to the whole domain occupied by the medium is taken as the volume V in (1.4).

2. Initial state of the shell. Let us assume that in the initial state of the body under consideration a Lagrangean coordinate system $\zeta^0 = \zeta$, ζ^1, ζ^2 can be selected such that the functions $x^i = r_0^i(\zeta, \zeta^\alpha)$ take the form (*)

$$r_0^i(\zeta, \zeta^\alpha) = x_0^i(\zeta^\alpha) + \zeta n_0^i(\zeta^\alpha) \tag{2.1}$$

where $x_0^i(\zeta^\alpha)$ are functions giving the middle surface Ω_0, n_0^i is the unit normal vector to Ω_0, $-h/2 \leqslant \zeta \leqslant h/2$, $h = h(\zeta^\alpha)$. The lower case Latin letters run through the values 0, 1, 2, and the lower case Greek letters through 1, 2. Expressions can be

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*) This assumption results in some constraints on the surface curvature Ω_0 and shell thickness $h(\zeta^\alpha)$, in particular, ribs are excluded on Ω_0. Such special cases must be examined separately.

obtained from (2.1) for the covariant metric tensor components in the initial state

$$g_{(0)\alpha\beta} = a_{0\alpha\beta} - 2\zeta b_{0\alpha\beta} + \zeta^2 c_{0\alpha\beta} \equiv (1 - K_0\zeta^2)\, a_{0\alpha\beta} - - 2\zeta(1 - H_0\zeta)\, b_{0\alpha\beta}; \qquad g_{(0)0\alpha} = 0, \qquad g_{(0)00} = 1 \tag{2.2}$$

Here $a_{0\alpha\beta}$, $b_{0\alpha\beta}$ and $c_{0\alpha\beta}$ are the coefficients of the first, second and third quadratic forms of the middle surface Ω_0, H_0 and K_0 are the mean and total curvature of Ω_0, respectively

$$H_0 = {}^1/_2 a_0^{\alpha\beta} b_{0\alpha\beta}, \qquad K_0 = det \,\|b_{0\alpha\beta}\| \,/\, det \,\|a_{0\alpha\beta}\|$$

The determinant of the metric tensor $g_{(0)\alpha\beta}$ must be known in order to evaluate the integrals over the initial volume. We find from (2.2)

$$\varkappa = \frac{g_0}{a_0} = (1 - 2H_0\zeta + K_0\zeta^2)^2 \tag{2.3}$$

$$g_0 = det \,\|g_{(0)ij}\|, \qquad a_0 = det \,\|a_{0\alpha\beta}\|$$

3. Deformed state of a shell. The radius vector of points of a body in the deformed state can always be represented as

$$r^i(\zeta, \zeta^\alpha) = x^i(\zeta^\alpha) + fn^i + f^\alpha x_\alpha{}^i \tag{3.1}$$

where $x^i(\zeta^\alpha)$ is the radius-vector of points of the middle surface Ω in the deformed state, $n^i(\zeta^\alpha)$ is the unit vector normal to Ω, $x_\alpha{}^i = \partial x^i(\zeta^\beta) \,/\, d\zeta^\alpha$ are tangential vectors to Ω. The vector $fn^i + f^\alpha x_\alpha{}^i$ is the radius-vector of points of the fibers (ζ, ζ^α) (ζ^α are fixed, and ζ a parameter along the fiber) relative to points with the coordinates $(0, \zeta^\alpha)$. In particular, if $f^\alpha = 0$, then the fiber in the deformed state remains perpendicular to the middle surface.

It is natural to assume that the dependence of the functions f and f^α on ζ in the internal part of the shell is determined by a finite number of parameters in the limit as the shell thickness $h \to 0$. In particular, it will be shown in Sect. 8 that the static Kirchhoff theory corresponds to the case when the first two terms are retained in the Taylor series expansion of the function f : (*)

$$f = (1 + e)\zeta + {}^1/_2\chi\zeta^2 \tag{3.2}$$

and the functions f^α are considered zero. Further we assume that f and f^α are known functions of ζ containing a finite number of free parameters, the internal degrees of

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*) If $x^i(\zeta^\alpha)$ are components of the radius-vector of points of the middle surface in the deformed state, then the functions f and f^α should, as follows from (3.1), satisfy the conditions

$$f(0, \zeta^\alpha) = 0, \qquad f^\alpha(0, \zeta^\alpha) = 0$$

However, $x^i(\zeta^\alpha)$ could be defined by other methods. For example, it can be assumed that

$$x^i(\zeta^\alpha) = \frac{1}{h} \int_{-h/2}^{h/2} r^i(\zeta, \zeta^\alpha)\, d\zeta$$

Then the integral constraints

$$\int_{-h/2}^{h/2} f(\zeta, \zeta^\alpha)\, d\zeta = 0, \qquad \int_{-h/2}^{h/2} f^\alpha(\zeta, \zeta^\alpha)\, d\zeta = 0$$

should be satisfied in selecting the dependence of the functions f and f^α on ζ.

freedom which we denote by $\mu^A(\zeta^\alpha)$

$$f = f(\zeta, \mu^A), \qquad f^\alpha = f^\alpha(\zeta, \mu^A) \tag{3.3}$$

Such parameters in the case of (3.2) are the quantities e and χ. Knowing the dependences of f and f^α on ζ and the parameters μ^A, the components of the metric tensor in the deformed state can be calculated by means of (1.5):

$$\begin{aligned} g_{\wedge\alpha\beta} &= a_{\alpha\beta} - 2fb_{\alpha\beta} + f^2 c_{\alpha\beta} + \frac{\partial f}{\partial \zeta^\alpha}\frac{\partial f}{\partial \zeta^\beta} + 2\nabla_{(\alpha}{}^{\wedge} f_{\beta)} + \\ &\nabla_\alpha{}^{\wedge} f^\gamma \nabla_\beta{}^{\wedge} f_\gamma - 2fb_{(\alpha}{}^{\gamma}\nabla_{\beta)}{}^{\wedge} f_\gamma + 2\frac{\partial f}{\partial \zeta}{}_{(\alpha}b_{\beta)\gamma}f^\gamma + b_{\alpha\gamma}b_{\beta\sigma}f^\gamma f^\sigma \\ g_{\wedge 0\alpha} &= \frac{\partial f_\alpha}{\partial \zeta} + \frac{\partial f}{\partial \zeta}\frac{\partial f}{\partial \zeta^\alpha} + \frac{\partial f_\gamma}{\partial \zeta}\nabla_\alpha{}^{\wedge} f^\gamma + \left(-f\frac{\partial f^\gamma}{\partial \zeta} + \frac{\partial f}{\partial \zeta}f^\gamma\right)b_{\alpha\gamma} \\ g_{\wedge 00} &= \left(\frac{\partial f}{\partial \zeta}\right)^2 + \frac{\partial f_\gamma}{\partial \zeta}\frac{\partial f^\gamma}{\partial \zeta}, \qquad f_\alpha = a_{\alpha\beta}f^\beta \end{aligned} \tag{3.4}$$

Here $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $c_{\alpha\beta}$ are coefficients of the first, second and third quadratic forms of the deformed surface, $\nabla_\alpha^{\wedge}$ is the covariant derivative with respect to the connectedness in Ω.

It is seen from (1.5), (3.3), (3.4) and (2.2) that the strain tensor components are known functions of the first, second and third strain tensors of the middle surface (*)

$$A_{\alpha\beta} = {}^1/_2(a_{\alpha\beta} - a_{0\alpha\beta}), \qquad B_{\alpha\beta} = b_{\alpha\beta} - b_{0\alpha\beta}, \qquad C_{\alpha\beta} = {}^1/_2(c_{\alpha\beta} - c_{0\alpha\beta}) \tag{3.5}$$

as well as ζ, μ^A and $\nabla_\alpha{}^{\wedge}\mu^A$

$$\varepsilon_{ij} = \varepsilon_{ij}(\zeta, A_{\alpha\beta}, B_{\alpha\beta}, \mu^A, \nabla_\alpha{}^{\wedge}\mu^A) \tag{3.6}$$

The functions (3.6) are easily written down in general form, however, it is more convenient to obtain them again every time in constructing specific models of shells.

Let us determine on which quantities the components of the shell particle velocity vector $w^{\cdot i}(\zeta, \zeta^\alpha, t)$ depend under the assumptions (3.3). Differentiating (3.1) with respect to time (the time was a parameter in all the preceding formulas in Sect. 3), we obtain

$$w^{\cdot i} \equiv \frac{\partial r^i}{\partial t}\bigg|_{\zeta^i=\text{const}} = v^i + (f^\alpha\delta_k{}^i - fx^{i\alpha}n_k)\frac{\partial v^k}{\partial \zeta^\alpha} + \frac{\partial f}{\partial \mu^A}\frac{\partial \mu^A}{\partial t}n^i + \frac{\partial f^\alpha}{\partial \mu^A}\frac{\partial \mu^A}{\partial t}x_\alpha{}^i \tag{3.7}$$

where $v^i = \partial x^i / \partial t$ are vector velocity components of points of the middle surface. The easily provable relationship

$$n_{,t}{}^i \equiv \frac{\partial n^i}{\partial t} = -x^{i\alpha}n_k\frac{\partial v^k}{\partial \zeta^\alpha}, \qquad x^{i\alpha} = a^{\alpha\beta}x_\beta{}^i \tag{3.8}$$

was used in deriving (3.7). Thus, for given functions f and f^α the vector velocity components of points of the shell depend in a known manner on the following parameters(**):

$$w^{\cdot i} = w^{\cdot i}\left(\zeta, v^k, \frac{\partial v^k}{\partial \zeta^\alpha}, x_\alpha{}^k, \mu^A, \frac{\partial \mu^A}{\partial t}\right) \tag{3.9}$$

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*) The third strain tensor is expressed in terms of $A_{\alpha\beta}$, $B_{\alpha\beta}$, $a_{0\alpha\beta}$ and $b_{0\alpha\beta}$ by algebraic relationships.

**) The components of the normal vector are expressed in terms of $x_\alpha{}^i$ by algebraic relationships.

4. Averaging of the variational equation. The variational equation (1.1) in elasticity theory is considered in the class of all twice-differentiable functions $r^i(\zeta^i, t)$ (or $w^i = r^i - r_0{}^i$). To obtain the fundamental relations of the theory of shells, let us consider the variational equation (1.1) in the class of functions (3.1), (3.2) (*). Hence, the integral of the action

$$I = \int_t \int_V \Lambda \, d\tau \, dt$$

as well as δW and δW^* become functionals defined by the functions $x^i(\zeta^\alpha, t)$, $\mu^A(\zeta^\alpha, t)$ (or $u^i = x^i - x_0{}^i$, μ^A). Let us find the form of these functionals. Let us take domains which are the direct products $V = \Omega_0 \times \zeta$, where Ω_0 is any part of the middle surface with piecewise-smooth boundaries, and $|\zeta| \leqslant h/2$, as the domain V in (1.1).

Since

$$I = \int_t \int_V \Lambda \sqrt{g_0} \, d\zeta \, d\zeta^1 d\zeta^2 dt = \int_t \int_{\Omega_0} \int_{-h/2}^{h/2} \Lambda \sqrt{\varkappa} \, d\zeta \, d\sigma \, dt, \qquad d\sigma = \sqrt{a_0} \, d\zeta^1 d\zeta^2 \quad (4.1)$$

then

$$I(u^i, \mu^A) = \int_t \int_{\Omega_0} L \, d\sigma \, dt$$

where L is the Lagrangean averaged over the plate thickness

$$L = \int_{-h/2}^{h/2} \Lambda \sqrt{\varkappa} \, d\zeta$$

We take the difference between the kinetic and internal energies (1.2) as the Lagrangean. Then L is represented as the difference between the averaged kinetic and internal energies

$$L = K - \Phi, \quad K = \int_{-h/2}^{h/2} \rho_0 \frac{w^{\cdot 2}}{2} \sqrt{\varkappa} \, d\zeta, \quad \Phi = \int_{-h/2}^{h/2} \rho_0 U \sqrt{\varkappa} \, d\zeta \qquad (4.2)$$

Formulas (3.6), (3.9), (4.2) and (1.5) show that the averaged kinetic energy K is a function of the velocity vector components of the middle surface, their derivatives along the surface, the tangent vectors $x_\alpha{}^i$, and also μ^A, $\partial\mu^A/\partial t$, and the shell characteristics ρ_0, h

$$K = K\left(v^i, \frac{\partial v^i}{\partial \zeta^\alpha}, x_\alpha{}^i, \mu^A, \frac{\partial \mu^A}{\partial t}, \rho_0, h\right) \qquad (4.3)$$

while the averaged internal energy is a function of the first and second strain tensors of the middle surface, the parameters μ^A and their derivatives $\nabla_\alpha{}^\wedge \mu^A$, and the shell characteristics K^B (we include ρ_0 and h among the parameters K^B to cut down the writing)

$$\Phi = \Phi(A_{\alpha\beta}, B_{\alpha\beta}, \mu^A, \nabla_\alpha{}^\wedge \mu^A, K^B) \qquad (4.4)$$

The Lagrangean depends substantially on the second derivatives of the displacement vector of the middle surface (the argument $\partial v^i / \partial \zeta^\alpha$ in the kinetic energy, and the

*) Such a method of obtaining the equations of the theory of shells, the passage from the general class of functions to functions of a special kind, is substantially the Ritz method. It was used by Reissner to construct a refined model of the linear static theory of plate bending. However, Reissner used a variational principle whose extension to arbitrary dynamical physically and geometrically nonlinear models causes difficulties.

argument $B_{\alpha\beta}$ in the internal energy), as well as on the internal degrees of freedom μ^A and their first derivatives. Substituting (3.1) and (3.3) into (1.4) (taking account of the equalities $\delta r^i = \delta w^i$ and $\delta n^i = -x^{i\alpha} n_k (\partial \delta u^k / \partial \zeta^\alpha)$) and integrating with respect to ζ, we obtain the following averaged expression for δW:

$$\delta W = \int_t \int_{\Omega_0} \left(P_i \delta u^i + P_i^{\alpha} \frac{\partial \delta u^i}{\partial \zeta^\alpha} + P_A \delta \mu^A \right) d\sigma \, dt + \int_t \int_{\partial\Omega_0} \left(Q_i \delta u^i + M_i^{\alpha} \frac{\partial \delta u^i}{\partial \zeta^\alpha} + S_A \delta \mu^A \right) ds \, dt - \left[\int_{\Omega_0} \left(\langle I_i \rangle \delta u^i + I_i^{\alpha} \frac{\partial \delta u^i}{\partial \zeta^\alpha} + I_A \delta \mu^A \right) d\sigma \right]_{t_1}^{t_2} \tag{4.5}$$

where $\partial\Omega_0$ is the boundary of the surface Ω_0, and the coefficients of the variations are determined by the formulas

$$P_i = \{p_i \sqrt{\varkappa_1}\} \quad P_i^{\alpha} = \{p_k (-n_i x^{k\alpha} f + \delta_i^k f^\alpha) \sqrt{\varkappa_1}\}$$

$$P_A = \left\{ p_i \left(n^i \frac{\partial f}{\partial \mu^A} + x_\alpha^i \frac{\partial f^\alpha}{\partial \mu^A} \right) \sqrt{\varkappa_1} \right\}$$

$$Q_i = \int_{-h/2}^{h/2} p_i \sqrt{\varkappa_2} \, d\zeta, \quad M_i^{\alpha} = \int_{-h/2}^{h/2} p_k (-n_i x^{k\alpha} f + \delta_i^k f^\alpha) \sqrt{\varkappa_2} \, d\zeta \tag{4.6}$$

$$S_A = \int_{-h/2}^{h/2} p_i \left(n^i \frac{\partial f}{\partial \mu^A} + x_\alpha^i \frac{\partial f^\alpha}{\partial \mu^A} \right) \sqrt{\varkappa_2} \, d\zeta$$

$$\langle I_i \rangle = \int_{-h/2}^{h/2} I_i \sqrt{\varkappa} \, d\zeta, \quad I_i^{\alpha} = \int_{-h/2}^{h/2} I_k (-n_i x^{k\alpha} f + \delta_i^k f^\alpha) \sqrt{\varkappa} \, d\zeta$$

$$I_A = \int_{-h/2}^{h/2} I_k \left(n^k \frac{\partial f}{\partial \mu^A} + x_\alpha^k \frac{\partial f^\alpha}{\partial \mu^A} \right) \sqrt{\varkappa} \, d\zeta$$

$$\sqrt{\varkappa_1} = \sqrt{\frac{g_1}{a_0}}, \quad g_1 = \det \| g_{(1)\alpha\beta} \|, \quad g_{(1)\alpha\beta} = g_{(0)\alpha\beta} + \frac{\partial h}{\partial \zeta^\alpha} \frac{\partial h}{\partial \zeta^\beta}$$

$$\sqrt{\varkappa_2} = \sqrt{\frac{a_2}{a_{(0)2}}} (1 - 2 b_{0\,\alpha\beta} \tau^\alpha \tau^\beta \zeta + c_{0\,\alpha\beta} \zeta^2)$$

Here $\{A\}$ denotes the sum $A|_{\zeta=h/2} + A|_{\zeta=-h/2}$, a_2 is the determinant of the metric tensor of the surface $\partial\Omega \times \zeta$ in the deformed state ($\partial\Omega$ is the boundary of Ω), $a_{(0)2}$ is the determinant of the metric tensor on the surface $\partial\Omega_0 \times \zeta$ in the initial state, τ_α are components of the unit tangent vector to $\partial\Omega_0$. In the case of a constant-thickness shell $\varkappa_1$ agrees with the quantity $\varkappa$ defined by (2.3).

Let us examine the meaning of the quantities (4.6) in the particular case of linearized theory and under the assumption that the fibers remain perpendicular to the middle surface under strain, i.e. $f^\alpha = 0$. Within the scope of linearized theory only the first member $f \approx \zeta$ of the Taylor series expansion of f should remain in the products fp_i. Hence $fp_i \approx \zeta p_i$. Evidently, $P_i d\sigma$ is the sum of forces acting on a shell element $d\sigma \times \zeta$ and applied to the side surfaces $\zeta = \mp h/2$, $P_i^{\alpha} d\sigma$ is the moment of tangential forces acting on the side surfaces relative to the middle surface, multiplied by the normal vector n_i

to the middle surface, Q_i is the transverse force, and M_i^α is the moment, relative to the $\zeta = 0$ axis, of the external forces acting on the surface $\partial\Omega \times \zeta$ multiplied by n_i. $\langle I_i \rangle$ is the momentum of a shell element averaged over the thickness, I_i^α is the moment of momentum of a shell element multiplied by n_i. The meaning of the quantities P_A, S_A and I_A is related to the meaning of the parameters μ^A.

An expression for δW^* is obtained analogously to (4.5)

$$\delta W^* = \int_t \int_{\Omega_0} \left(P_i^* \delta u^i + P_i^{*\alpha} \frac{\partial \delta u^i}{\partial \zeta^\alpha} + P_A^* \delta\mu^A\right) d\sigma\, dt \tag{4.7}$$

where

$$P_i^* = \int_{-h/2}^{h/2} \rho_0 F_i \sqrt{\varkappa}\, d\zeta$$

$$P_i^{*\alpha} = \int_{-h/2}^{h/2} \rho_0 F_k \left(-n_i x^{k\alpha} f + \delta_i^k f^\alpha\right) \sqrt{\varkappa}\, d\zeta \tag{4.8}$$

$$P_A^* = \int_{-h/2}^{h/2} \rho_0 F_k \left(n^k \frac{\partial f}{\partial \mu^A} + x_\alpha^k \frac{\partial f^\alpha}{\partial \mu^A}\right) \sqrt{\varkappa}\, d\zeta$$

The quantity $P_i^* d\sigma$ has the meaning of a total mass force acting on an element $d\sigma \times \zeta$, $P_i^{*\alpha}$ (in the linearized theory for $f^\alpha = 0$) is the moment of external volume forces relative to the middle surface, multiplied by the unit normal vector n_i.

In conformity with (4.1), (4.2), (4.5) and (4.7), the averaged variational equation (1.1) becomes

$$\delta \int_t \int_{\Omega_0} (K - \Phi)\, d\sigma\, dt + \int_t \int_{\Omega_0} \Big[(P_i + P_i^*) \delta u^i + (P_i^\alpha + P_i^{*\alpha}) \frac{\partial \delta u^i}{\partial \zeta^\alpha} +$$
$$(P_A + P_A^*) \delta\mu^A \Big] d\sigma\, dt + \int_t \int_{\partial\Omega_0} \left(Q_i \delta u^i + M_i^\alpha \frac{\partial \delta u^i}{\partial \zeta^\alpha} + S_A \delta\mu^A\right) ds\, dt - \tag{4.9}$$
$$\left[\int_{\Omega_0} \left(\langle I_i \rangle \delta u^i + I_i^\alpha \frac{\partial \delta u^i}{\partial \zeta^\alpha} + I_A \delta\mu^A \right) d\sigma \right]_{t_1}^{t_2} = 0$$

5. System of equations of the theory of shells. Let us calculate the variation of the first member in (4.9) by considering that the functions μ^A in the domain of variation ζ^α, t are twice, and u^i fourfold differentiable

$$\delta \int_t \int_{\Omega_0} K\, d\sigma\, dt = \int_t \int_{\Omega_0} \left(\frac{\delta K}{\delta u^i} \delta u^i + \frac{\delta K}{\delta \mu^A} \delta\mu^A \right) d\sigma\, dt + \tag{5.1}$$
$$\int_t \int_{\partial\Omega_0} \left(\frac{\partial K}{\partial x_\alpha^{\ i}} - \frac{\partial}{\partial t} \frac{\partial K}{\partial v_\alpha^{\ i}} \right) \delta u^i \nu_\alpha\, ds\, dt + \left[\int_{\Omega_0} \left(\frac{\partial K}{\partial v^i} \delta u^i + \frac{\partial K}{\partial v_\alpha^{\ i}} \frac{\partial \delta u^i}{\partial \zeta^\alpha} + \frac{\partial K}{\partial \dot\mu^A} \delta\mu^A \right) d\sigma \right]_{t_1}^{t_2}$$

Here

$$\frac{\delta K}{\delta u^i} = -\frac{\partial}{\partial t} \frac{\partial K}{\partial v^i} + \frac{\partial}{\partial t} \nabla_\alpha^\circ \frac{\partial K}{\partial v_\alpha^{\ i}} - \nabla_\alpha^\circ \frac{\partial K}{\partial x_\alpha^{\ i}}, \quad v_\alpha^{\ i} \equiv \frac{\partial v^i}{\partial \zeta^\alpha} \tag{5.2}$$

$$\frac{\delta K}{\delta \mu^A} = \frac{\partial K}{\partial \mu^A} - \frac{\partial}{\partial t} \frac{\partial K}{\partial \dot\mu^A}, \quad \dot\mu^A \equiv \frac{\partial \mu^A}{\partial t}$$

ν_α are the components of the unit vector normal to the curve $\partial\Omega_0$, tangent to the surface Ω_0 and directed exterior to Ω_0, and ∇_α° is the covariant derivative with respect to the connectedness of the surface Ω_0.

Using the formulas (*)

$$\delta A_{\alpha\beta} = \frac{\partial \delta u_i}{\partial \zeta^{(\alpha}} x_{\beta)}{}^i, \qquad \delta B_{\alpha\beta} = n_i \nabla_\alpha{}^\wedge \nabla_\beta{}^\wedge \delta u^i \tag{5.3}$$

$$\nabla_\alpha{}^\wedge \varphi^\alpha = \frac{1}{\gamma} \nabla_\alpha{}^\circ \gamma \varphi^\alpha, \quad \nabla_\alpha{}^\circ \varphi^\alpha = \gamma \nabla_\alpha{}^\wedge \frac{1}{\gamma} \varphi^\alpha, \quad \gamma = \sqrt{\frac{a}{a_0}}, \quad a = \det \|a_{\alpha\beta}\|$$

for the variations of the internal energy, we obtain from (4.4)

$$\delta \int_t \int_{\Omega_0} \Phi \, d\sigma \, dt = \int_t \int_{\Omega_0} \left(\frac{\delta \Phi}{\delta u^i} \delta u^i + \frac{\delta \Phi}{\delta \mu^A} \delta \mu^A \right) d\sigma \, dt + \tag{5.4}$$

$$\int_t \int_{\partial\Omega_0} \gamma \left(n^{\alpha\beta} \delta u_\alpha - q^\beta \delta u_n + m^{\alpha\beta} n_i \frac{\partial \delta u^i}{\partial \zeta^\alpha} + \frac{\partial \Phi / \gamma}{\partial \mu_\beta{}^A} \delta \mu^A \right) \nu_\beta \, ds \, dt$$

Here

$$\frac{\delta \Phi}{\delta u^i} = n_i \gamma (\nabla_\beta{}^\wedge q^\beta - n^{\alpha\beta} b_{\alpha\beta}) - x_{i\alpha} \gamma (\nabla_\beta{}^\wedge n^{\alpha\beta} + q^\beta b_\beta{}^\alpha)$$

$$\frac{\delta \Phi}{\delta \mu^A} = \gamma \left(\frac{\partial \Phi / \gamma}{\partial \mu^A} - \nabla_\alpha{}^\wedge \frac{\partial \Phi / \gamma}{\partial \mu_\alpha{}^A} \right)$$

$$n^{\alpha\beta} = \frac{1}{\gamma} \frac{\partial \Phi}{\partial A_{\alpha\beta}} + m^{\gamma\beta} b_\gamma{}^\alpha, \qquad m^{\alpha\beta} = \frac{1}{\gamma} \frac{\partial \Phi}{\partial B_{\alpha\beta}} \tag{5.5}$$

$$q^\alpha = \nabla_\beta{}^\wedge m^{\alpha\beta} \tag{5.6}$$

Also $\mu_\alpha{}^A$ denotes the derivative $\nabla_\alpha{}^\wedge \mu^A$, δu_α and δu_n are the projections of variations of the displacement on the tangent and normal to Ω:

$$\delta u_\alpha = x_\alpha{}^i \delta u_i, \quad \delta u_n = n^i \delta u_i.$$

By integration by parts, the second member in (4.9) is reduced to the following:

$$\int_t \int_{\Omega_0} \left[(P_i + P_i{}^*) \delta u^i + (P_i{}^\alpha + P_i{}^{*\alpha}) \frac{\partial \delta u^i}{\partial \zeta^\alpha} + (P_A + P_A{}^*) \delta \mu^A \right] d\sigma \, dt =$$

$$\int_t \int_{\Omega_0} [(P_i + P_i{}^* - \nabla_\alpha{}^\circ (P_i{}^\alpha + P_i{}^{*\alpha})) \delta u^i + (P_A + P_A{}^*) \delta \mu^A] \, d\sigma \, dt +$$

$$\int_t \int_{\partial\Omega_0} (P_i{}^\alpha + P_i{}^{*\alpha}) \delta u^i \nu_\alpha \, ds \, dt \tag{5.7}$$

Substituting (5.1), (5.4) and (5.7) into (4.9) and first assuming the variations δu^i and $\delta \mu^A$ to be zero on the boundary of the three-dimensional domain $\Omega_0 \times t$, we obtain the equations of the theory of shells

$$\frac{\delta K}{\delta u^i} - \frac{\delta \Phi}{\partial u^i} + P_i + P_i{}^* - \nabla_\alpha{}^\circ (P_i{}^\alpha + P_i{}^{*\alpha}) = 0 \tag{5.8}$$

$$\frac{\delta K}{\delta \mu^A} - \frac{\delta \Phi}{\delta \mu^A} + P_A + P_A{}^* = 0 \tag{5.9}$$

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*) The parentheses on the indices denote the operation of symmetrization; φ^α in (5.3) are contravariant components of any two-dimensional vector in the ζ^α coordinate system.

The quantities in these equations are defined by (4.6), (4.8), (5.2) and (5.5). Equations (5.8) are the equations of shell motion, (5.9) are used to determine the degrees of freedom μ^A. The equations of motion take on an especially simple form in the static case (the kinetic energy is $K = 0$). Projecting (5.8) onto the normal and tangent plane to Ω and using (5.5), we obtain

$$\nabla_\beta{}^\wedge q^\beta - n^{\alpha\beta} b_{\alpha\beta} = \gamma^{-1} n^i \left[P_i + P_i^* - \nabla_\alpha{}^\circ (P_i^\alpha + P_i^{*\alpha})\right] \tag{5.10}$$

$$\nabla_\beta{}^\wedge n^{\alpha\beta} + q^\beta b_\beta{}^\alpha = \gamma^{-1} x^{i\alpha} \left[P_i + P_i^* - \nabla_\alpha{}^\circ (P_i^\alpha + P_i^{*\alpha})\right] \tag{5.11}$$

The members in the left sides of (5.10), (5.11) agree outwardly with the corresponding terms in the equilibrium equations ordinarily used in the theory of shells. In the right sides of (5.10), (5.11) are written what should be understood to be the external forces acting on a shell element. Let us emphasize that the tangential forces on the surface $\zeta = \pm h/2$ yield a contribution to (5.10) which is the projection of the equilibrium equations in the normal direction to the middle surface. This contribution is described by the tensor P_i^α, (see (4.6)). The tensile force tensor $n^{\alpha\beta}$ and the bending moment tensor $m^{\alpha\beta}$ are given by the equations of state (5.5). The moment equations, which are usually appended to the system of equilibrium equations (5.10), (5.11), are the definition of the quantities q^α (5.6) in the theory under consideration.

In the dynamic case (kinetic energy $K \neq 0$), the projections of the variational derivative $\gamma^{-1}\ \delta K / \delta u^i$ on the normal and on the tangent planes to Ω, i.e. $\gamma^{-1} n^i \delta K / \delta u^i$ and $-\gamma^{-1} x_\alpha{}^i \delta K / \delta u^i$, should be added to the right sides of (5.10), (5.11). The dynamic equations are simplified for shell models in which it is assumed that the normal to the middle surface goes over, under deformation, into a normal (the functions f^α in (3.1) equal to zero). As is seen from (3.7) and (4.2), the kinetic energy depends on $x_\alpha{}^i$ and the velocity gradients $\partial v^i / \partial \zeta^\alpha$ only on terms of the combinations

$$a_{\alpha\beta} = g_{ij} x_\alpha{}^i x_\beta{}^j, \qquad n_{,t}{}^i = \frac{\partial n^i}{\partial t} = - x^{i\alpha} n_k \frac{\partial v^k}{\partial \zeta^\alpha}$$

The variational derivative of K becomes

$$\frac{\delta K}{\partial u^i} = - \frac{\partial}{\partial t} \frac{\partial K}{\partial v^i} - \gamma \nabla_\beta{}^\wedge \frac{1}{\gamma} \left(x^{k\beta} n_i \frac{\partial}{\partial t} \frac{\partial K}{\partial n_{,t}{}^k} + 2 x_{i\alpha} \frac{\partial K}{\partial a_{\alpha\beta}} \right) \tag{5.12}$$

Substituting (5.12) into (5.8) and introducing the notation

$$N^\beta = \nabla_\gamma{}^\wedge m^{\beta\gamma} + \frac{1}{\gamma} x^{k\beta} \frac{\partial}{\partial t} \frac{\partial K}{\partial n_{,t}{}^k} \tag{5.13}$$

we write the dynamic equations as follows:

$$\nabla_\beta{}^\wedge N^\beta - \left(n^{\alpha\beta} - \frac{2}{\gamma} \frac{\partial K}{\partial a_{\alpha\beta}} \right) b_{\alpha\beta} + \frac{1}{\gamma} n^i \frac{\partial}{\partial t} \frac{\partial K}{\partial v^i} =$$
$$\frac{1}{\gamma} n^i \left[P_i + P_i^* - \nabla_\alpha{}^\circ (P_i^\alpha + P_i^{*\alpha})\right] \tag{5.14}$$

$$\nabla_\beta{}^\wedge \left(n^{\alpha\beta} - \frac{2}{\gamma} \frac{\partial K}{\partial a_{\alpha\beta}} \right) + N^\beta b_\beta{}^\alpha - \frac{1}{\gamma} x^{k\alpha} \frac{\partial}{\partial t} \frac{\partial K}{\partial v^k} =$$
$$-\frac{1}{\gamma} x^{i\alpha} \left[P_i + P_i^* - \nabla_\alpha{}^\circ (P_i^\alpha + P_i^{*\alpha})\right] \tag{5.15}$$

The relationship (5.13), which is the definition of N^β, can be considered, as before, as the equation which replaces the equation of the moment of momentum. Projections of the derivative of the momentum on the normal and tangent with respect to time as well as the dynamical addition to $n^{\alpha\beta}$ (the second term in parentheses on the left side; this term can be essential only in nonlinear theories) appeared in the equations of motion (5.14), (5.15).

Let us consider the additional relationships which can be extracted from the variational equation (4.9). For arbitrary variations δu^i and $\delta\mu^A$ on the boundary of the $V \times t$ domain (4.9) reduces to the following:

$$\int_t\int_{\partial\Omega_0}\left\{\left[\left(\frac{\partial K}{\partial x_\alpha{}^i}-\frac{\partial}{\partial t}\frac{\partial K}{\partial v_\alpha{}^i}-\gamma n^{\beta\alpha}x_{i\beta}+\gamma q^\alpha n_i+P_i{}^\alpha+P_i{}^{*\alpha}\right)\nu_\alpha+Q_i\right]\delta u^i+\right.$$
$$\left.(M_i{}^\alpha-\gamma m^{\alpha\beta}\nu_\beta n_i)\frac{\partial\delta u^i}{\partial\zeta^\alpha}+\left(S_A-\frac{\partial\Phi}{\partial\mu_\alpha{}^A}\nu^\alpha\right)\delta\mu^A\right\}ds\,dt+\qquad(5.16)$$
$$\left[\int_\Omega\left\{\left(\frac{\partial K}{\partial v^i}-\langle I_i\rangle\right)\delta u^i+\left(\frac{\partial K}{\partial v_\alpha{}^i}-I_i{}^\alpha\right)\frac{\partial\delta u^i}{\partial\zeta^\alpha}+\left(\frac{\partial K}{\partial\dot\mu^A}-I_A\right)\delta\mu^A\right\}d\sigma\right]_{t_1}^{t_2}=0$$

This equality can be satisfied if it is assumed that

$$Q_i=\left(\gamma n^{\beta\alpha}x_{i\beta}-\gamma q^\alpha n_i-P_i{}^\alpha-P_i{}^{*\alpha}+\frac{\partial}{\partial t}\frac{\partial K}{\partial v_\alpha{}^i}-\frac{\partial K}{\partial x_\alpha{}^i}\right)\nu_\alpha$$
$$M_i{}^\alpha=\gamma m^{\alpha\beta}\nu_\beta n_i,\qquad S_A=\frac{\partial\Phi}{\partial\mu_\alpha{}^A}\nu_\alpha\qquad(5.17)$$
$$\langle I_i\rangle=\frac{\partial K}{\partial v^i},\qquad I_i{}^\alpha=\frac{\partial K}{\partial v_\alpha{}^i},\qquad I_A=\frac{\partial K}{\partial\dot\mu^A}$$

The equality (5.16) is also satisfied for other values of Q_i, $M_i{}^\alpha$, ... (this question has been considered in [2] for arbitrary models of continuous media with high derivatives). However, in all the relationships used later, the quantities Q_i, $M_i{}^\alpha$, ... enter in combinations for which the existing arbitrariness is immaterial. Formulas (5.17) can be considered as the definitions of the quantities Q_i, ..., I_A.

6. Boundary conditions. As in general theory of models of continuous media, let us give the boundary conditions by specifying the functional

$$\delta A^{(e)}=\int_t\int_{\partial\Omega_0}\left(Q_i\delta u^i+M_i{}^\alpha\frac{\partial\delta u^i}{\partial\zeta^\alpha}+S_A\delta\mu^A\right)ds\,dt\qquad(6.1)$$

which is the work of the external forces on the possible displacements δu^i and $\delta\mu^A$ on the shell boundary. The external forces do work not only on the displacements δu^i of the shell edge, but also on the gradient of the displacements $\partial\delta u^i/\partial\zeta^\alpha$. The work on the gradients of the displacements is performed, in conformity with (5.17), by the bending moments

$$\int_t\int_{\partial\Omega_0}M_i{}^\alpha\frac{\partial\delta u^i}{\partial\zeta^\alpha}ds\,dt=\int_t\int_{\partial\Omega_0}\gamma m^{\alpha\beta}\nu_\beta n_i\frac{\partial\delta u^i}{\partial\zeta^\alpha}ds\,dt\qquad(6.2)$$

It is convenient to decompose the gradient of the displacements $\partial\delta u^i/\partial\zeta^\alpha$ in (6.2) into the derivative along the normal to $\partial\Omega_0$ and into the derivative along $\partial\Omega_0$

$$\frac{\partial\delta u^i}{\partial\zeta^\alpha}=\nu_\alpha\left(\nu^\beta\frac{\partial\delta u^i}{\partial\zeta^\beta}\right)+\tau_\alpha\left(\tau^\beta\frac{\partial\delta u^i}{\partial\zeta^\beta}\right)=\nu_\alpha\frac{\partial\delta u^i}{\partial\nu}+\tau_\alpha\frac{d\delta u^i}{ds}$$

Then the work of the bending moments is represented as

$$\int_t \int_{\partial\Omega_0} M_i^{\ \alpha} \frac{\partial \delta u^i}{\partial \zeta^\alpha} \, ds\, dt = \int_t \int_{\partial\Omega_0} \gamma \left(m^{\alpha\beta} \nu_\alpha \nu_\beta n_i \frac{\partial \delta u^i}{\partial \nu} + m^{\alpha\beta} \tau_\alpha \nu_\beta n_i \frac{d\delta u^i}{ds} \right) ds\, dt \quad (6.3)$$

The first member is the work of the bending moments, and the second one of the torques. We represent the total work of the external forces on the possible displacements by using (6.3) and integrating by parts, as

$$\delta A^{(e)} = \int_t \int_{\partial\Omega_0} \left\{ \left(Q_i - \frac{d}{ds} \gamma m^{\alpha\beta} \tau_\alpha \nu_\beta n_i \right) \delta u^i + \gamma m^{\alpha\beta} \nu_\alpha \nu_\beta n_i \frac{\partial \delta u^i}{\partial \nu} + S_A \delta \mu^A \right\} ds\, dt + \int_t \int_{\partial\Omega_0} \frac{d}{ds} (\gamma m^{\alpha\beta} \tau_\alpha \nu_\beta n_i \delta u^i)\, ds\, dt \quad (6.4)$$

The last integral in (6.4) is zero if there are no points of discontinuity in the quantity $\gamma m^{\alpha\beta}\tau_\alpha \nu_\beta n_i \delta u^i$ on the contour $\partial\Omega_0$. The procedure of integrating by parts is the replacement of the system of torques by an equivalent transverse force. The expression (6.4) for $\delta A^{(e)}$ possesses the advantage that the variations δu^i and $\partial \delta u^i / \partial \nu$ are independent.

Let us prescribe the work of the external forces on the shell boundary

$$\delta A^{(e)} = \int_t \int_{\partial\Omega_0} \left(\mathbf{Q}_i \delta u^i + \mathbf{M} n_i \frac{\partial \delta u^i}{\partial \nu} + \mathbf{S}_A \delta \mu^A \right) ds\, dt \quad (6.5)$$

Here $\mathbf{Q}_i$ is the external force applied to the shell edge, $\mathbf{M}$ is the external bending moment. Various boundary conditions can be obtained from (6.4) and (6.5) depending on the construction of the function class. For example, if displacements are given on the boundary, then $\delta u^i = 0$ and by virtue of the arbitrariness of $n_i\, \partial \delta u^i / \partial \nu$ and $\delta \mu^A$

$$m^{\alpha\beta} \nu_\alpha \nu_\beta = \mathbf{M}, \qquad S_A = \mathbf{S}_A \quad (6.6)$$

When there are no constraints on the displacements on the boundary, then by virtue of the independence of δu^i and $\partial \delta u^i / \partial \nu$ we obtain from (6.4) and (6.5) in addition to (6.6)

$$Q_i - \frac{d}{ds} (\gamma m^{\alpha\beta} \tau_\alpha \nu_\beta n_i) = \mathbf{Q}_i \quad (6.7)$$

If Q_i and $m^{\alpha\beta}$ are defined in terms of the internal and kinetic energies by (5.6), (5.17), then the left side of (6.7) can be considered as the internal stress resultants originating to equilibrate the external forces $\mathbf{Q}_i$. It is essential that these internal stress resultants depend on the curvature of the middle surface and the curvature of its boundary.

As is seen from a comparison between (6.4) and (6.5), the last integral in (6.4) vanishes and in the presence of points of discontinuity M_s of the quantities $\gamma m^{\alpha\beta}\tau_\alpha \nu_\beta n_i$ on the boundary. We obtain the following relationships:

$$(\gamma m^{\alpha\beta} \nu_\alpha \tau_\beta n_i)_+ = (\gamma m^{\alpha\beta} \nu_\alpha \tau_\beta n_i)_- \quad (6.8)$$

at the points M_s for continuous variations δu_i on $\partial\Omega_0$ from the equality of this integral to zero. The plus and minus symbols here denote the limit values of the quantities upon approaching M_s from the right and left along $\partial\Omega_0$. The equality (6.8) means that if the normal vector to the middle surface is continuous in the deformed state, then the magnitude of the torque should be continuous (in linearized theories $\gamma = 1$).

The equality (6.8) makes sense only when the appropriate limits $(\gamma m^{\alpha\beta}\tau_\alpha \nu_\beta n_i)_\pm$

are defined. The conditions at the singular points at which there are singulatities can also be obtained by variational methods, and even the nature of the singularity can be determined, however, this is a topic of a separate investigation.

7. Conditions on lines of discontinuity. On the surface Ω_0 let there be a line Γ on which the derivatives of the displacements and the parameters μ^A can be discontinuous. The line Γ generally moves along Ω_0. Let us establish the conditions which should be satisfied on the lines of discontinuity.

A moving line Γ outlines a two-dimensional surface

$$\zeta^\alpha = F^\alpha(s, t) \tag{7.1}$$

in the three-dimensional space of the variables ζ^1, ζ^2, t. Without limiting the generality, it can be assumed that this surface separates the domain $\Omega_0 \times t$ into two parts. Let quantities in one of them be denoted by the subscript 1, and in the other by 2. The quantity of conditions on the discontinuity depends on the construction of the function class in the neighborhood of the surface of discontinuity $\Gamma \times t$. The admissible functions u^i and μ^A in each of the domains 1 and 2 have fourth and second derivatives, respectively, by assumption. Let us consider the components u^i of the displacement vector to be continuous in $\Gamma \times t$. Assumptions relative to μ^A and the derivatives of u^i are made below.

There are two possibilities for the functions (7.1): (1) The functions (7.1) are specified, the motion of the line of discontinuity is known, and the admissible functions undergo a discontinuity on a surface fixed in advance, and (2) the motion of the line of discontinuity is to be determined, the admissible functions may discontinue on any (no longer fixable) surface $\Gamma \times t$, therefore the surface $\Gamma \times t$ itself (the function (7.1)) is also subject to variation together with u^i and μ^A.

Let us first consider the case (1) when the motion of the line of discontinuity is known. The Stokes formula which was used to evaluate the variation

$$\delta \int_t \int_\Omega (K - \Phi)\, d\sigma\, dt \tag{7.2}$$

for the integration by parts, is complicated for functions having discontinuities, and is written as

$$\int_t \int_\Omega \left(\nabla_\alpha \Phi^\alpha + \frac{\partial A}{\partial t}\right) d\sigma\, dt = \int_t \int_{\partial\Omega_0} \Phi^\alpha \nu_\alpha\, ds\, dt +$$

$$\left[\int_{\Omega_0} A\, d\sigma\right]_{t_1}^{t_2} + \int_t \int_\Gamma (c\,[A] + [\Phi^\alpha]\,\nu_\alpha)\, ds\, dt \tag{7.3}$$

$$[A] = A_2 - A_1, \qquad c = -\nu_\alpha \frac{\partial F^\alpha}{\partial t}$$

where A and Φ_α are arbitrary functions undergoing discontinuities on $\Gamma \times t$, c is the velocity of the motion of the line of discontinuity along its normal, and by assumption the unit vector ν_α of the normal to Γ is directed from the side 2 to the side 1. The addition

$$\int_t \int_\Gamma \Big\{ \Big[\frac{\partial K}{\partial v^i} \delta u^i + \frac{\partial K}{\partial v_\alpha{}^i} \frac{\partial \delta u^i}{\partial \zeta^\alpha} + \frac{\partial K}{\partial \mu^{\cdot A}} \delta \mu^A \Big] c + \Big[\Big(\frac{\partial K}{\partial x_\alpha{}^i} - \frac{\partial}{\partial t} \frac{\partial K}{\partial v_\alpha{}^i} \Big) \delta u^i \Big| \nu_\alpha -$$

$$- \left[\gamma \left(n^{\alpha\beta} x_{i\alpha} - q^{\beta} n_i \right) \delta u^i + \gamma m^{\alpha\beta} n_i \frac{\partial \delta u^i}{\partial \zeta^\alpha} + \frac{\partial \Phi}{\partial \mu_\alpha{}^A} \delta \mu^A \right] \nu_\alpha \Big\} ds\, dt = 0 \quad (7.4)$$

appears in the expression for the variation (7.2) in conformity with (7.3). The fact that this addition is zero follows from the variational equation (4.9) and the independence of the variations on $\partial\Omega_0$ and Γ. Besides the functions u^i let $\partial u^i / \partial \zeta^\alpha$, μ^A also be continuous. Then

$$[\delta u^i] = 0, \qquad \left[\frac{\partial \delta u^i}{\partial \zeta^\alpha} \right] = 0 \qquad [\delta \mu^A] = 0$$

Hence, (7.4) reduces to

$$\int_t \int_\Gamma \Big\{ \Big(\left[\frac{\partial K}{\partial v^i} \right] c + \left[\frac{\partial K}{\partial x_\alpha{}^i} - \frac{\partial}{\partial t} \frac{\partial K}{\partial v_\alpha{}^i} \right] \nu_\alpha - [\gamma n^{\alpha\beta} x_{i\alpha} - \gamma q^\beta n_i] \nu_\beta \Big) \delta u^i +$$

$$\Big(\left[\frac{\partial K}{\partial v_\alpha{}^i} \right] c - [\gamma m^{\alpha\beta} n_i] \nu_\beta \Big) \frac{\partial \delta u^i}{\partial \zeta^\alpha} + \Big(\left[\frac{\partial K}{\partial \mu^{\cdot A}} \right] c - \left[\frac{\partial \Phi}{\partial \mu_\alpha{}^A} \right] \nu_\alpha \Big) \delta \mu^A \Big\} ds\, dt = 0$$

Extracting the independent variations from this relationship by integration by parts (analogously to Sect. 6), we obtain the following conditions on the discontinuity:

$$\left[\frac{\partial K}{\partial v^i} \right] c + \left[\frac{\partial K}{\partial x_\alpha{}^i} - \frac{\partial}{\partial t} \frac{\partial K}{\partial v_\alpha{}^i} \right] \nu_\alpha - [\gamma n^{\alpha\beta} x_{i\alpha} - \gamma q^\beta n_i] \nu_\beta - \quad (7.5)$$

$$\frac{d}{ds} \tau_\alpha \Big(\left[\frac{\partial K}{\partial v_\alpha{}^i} \right] c - [\gamma m^{\alpha\beta} n_i] \nu_\beta \Big) = 0$$

$$\left[\frac{\partial K}{\partial v_\alpha{}^i} \right] c \nu_\alpha - [\gamma m^{\alpha\beta} n_i] \nu_\alpha \nu_\beta = 0 \quad (7.6)$$

$$\left[\frac{\partial K}{\partial \mu^{\cdot A}} \right] c - \left[\frac{\partial \Phi}{\partial \mu_\alpha{}^A} \right] \nu_\alpha = 0 \quad (7.7)$$

If a discontinuity of the derivatives of the displacements $u_\nu{}^i = \nu^\alpha (\partial u^i / \partial \zeta^\alpha)$ along the normal is assumed, then the variations $\delta u_\nu{}^i$ become independent on both sides of the surface $\Gamma \times t$, and we obtain from (7.4) in place of (7.6), that the combination

$$\frac{\partial K}{\partial v_\alpha{}^i} \nu_\alpha c - \gamma m^{\alpha\beta} n_i \nu_\alpha \nu_\beta = 0 \quad (7.8)$$

vanishes on each side of $\Gamma \times t$. Analogously, in the case of a discontinuity of μ^A on $\Gamma \times t$, the quantities

$$\frac{\partial K}{\partial \mu^{\cdot A}} c - \frac{\partial \Phi}{\partial \mu_\alpha{}^A} \nu_\alpha = 0 \quad (7.9)$$

vanish on each side of the surface $\Gamma \times t$.

Now, let the motion of the line of discontinuity not be known, and let it require to find the solution of the problem as a result. To do this, additional conditions are necessary. We obtain the appropriate conditions from the variational equation (4.9) by considering it in the class of all functions undergoing a discontinuity on some (no longer fixable) surface $\Gamma \times t$. The surface $\Gamma \times t$ itself is hence subject to variation together with the functions u^i and μ^A.

It is seen that the variational equation (4.9) reduces to the equality (7.3) in which the member

$$\int_t \int_\Gamma [K - \Phi] \nu_\alpha \delta \zeta^\alpha \, ds\, dt$$

is added in the left side, where $\delta\zeta^\alpha$ is the variation of the functions (7.1), and δu^i and

$\delta\mu^A$ denote a variation of the form of the functions u^i and μ^A.

Let us consider that u^i, $u_{,\alpha}{}^i \equiv \partial u^i / \partial\zeta^\alpha$, μ^A are continuous on $\Gamma \times t$. In this case the total variations

$$\delta_\Pi u^i = \delta u^i + u^i_{,\alpha}\delta\zeta^\alpha,\ \delta_\Pi u^i_{,\alpha} = \delta u^i_{,\alpha} + \nabla_\beta{}^\circ u^i_{,\alpha}\delta\zeta^\beta,\ \delta_\Pi\mu^A = \delta\mu^A + \nabla_\alpha{}^\circ\mu^A\delta\zeta^\alpha$$

are continuous on $\Gamma \times t$

$$[\delta_\Pi u^i] = 0, \qquad [\delta_\Pi u^i_{,\alpha}] = 0, \qquad [\delta_\Pi\mu^A] = 0$$

First assuming $\delta\zeta^\alpha = 0$ (hence $\delta_\Pi = \delta$), we obtain (7.5) – (7.7) from (7.3). Relationships which reduce to the following energy condition when using the kinematic conditions:

$$[\Phi - K] + \left(\frac{\partial K}{\partial v_\alpha{}^i} \nu^\alpha c - \gamma m^{\alpha\beta} n_i \nu_\alpha \nu_\beta\right)\omega^i + \left(\frac{\partial K}{\partial \dot\mu^A} c - \frac{\partial\Phi}{\partial\mu_\alpha{}^A}\nu^\alpha\right)\omega^A = 0 \qquad (7.10)$$

$$(\omega^i = [\nabla_\beta{}^\circ u^i_{,\alpha}]\nu^\alpha\nu^\beta,\ \omega^A = [\nabla_\alpha{}^\circ\mu^A]\nu^\alpha)$$

follow from (7.3) for arbitrary $\delta\zeta^\alpha$.

If the derivatives $\partial u^i / \partial\zeta^\alpha$ are discontinuous on $\Gamma \times t$ then we obtain (7.8) from (7.3) instead of (7.6), and the following relation instead of (7.10):

$$[\Phi - K] + \left\{\frac{\partial K}{\partial v^i} c + \left(\frac{\partial K}{\partial x_\alpha{}^i} - \frac{\partial}{\partial t}\frac{\partial K}{\partial v_\alpha{}^i}\right)\nu_\alpha - \gamma n^{\alpha\beta} x_{i\alpha}\nu_\beta + \gamma q^\beta \nu_\beta n_i - \right.$$

$$\left.\frac{d}{ds}\left(\frac{\partial K}{\partial v_\alpha{}^i} c - \gamma m^{\alpha\beta}\nu_\beta n_i\right)\tau_\alpha\right\}\theta^i + \left(\frac{\partial K}{\partial\dot\mu^A} c - \frac{\partial\Phi}{\partial\mu_\alpha{}^A}\nu^\alpha\right)\omega^A = 0 \qquad (7.11)$$

$$(\theta^i = [u^i_{,\alpha}]\nu^\alpha)$$

If the parameters μ^A are also discontinuous on $\Gamma \times t$, then condition (7.7) is replaced by (7.9), and the additional energy relationship has the form (7.10) or (7.11) (for $\theta^i = 0$ or $\theta^i \neq 0$). Hence, the last member in (7.10) vanishes by virtue of (7.9).

8. Illustration. Linear statics of isotropic plates. Let us consider physically and geometrically linear models of isotropic plates for which the internal energy ρU is

$$2\rho U = \lambda(\varepsilon_i{}^i)^2 + 2\mu\varepsilon_{ij}\varepsilon^{ij} \qquad (8.1)$$

Because of the geometric linearity of the theory, (1.5) and (3.4) simplify ($f - \zeta$, f^α, $A_{\alpha\beta}$, $B_{\alpha\beta}$, $C_{\alpha\beta}$ in (3.4) should be considered small), and result in the following relationships:

$$\varepsilon_{\alpha\beta} = A_{\alpha\beta} - \zeta B_{\alpha\beta} + \nabla_{(\alpha}f_{\beta)} \qquad (8.2)$$

$$\varepsilon_{0\alpha} = \frac{1}{2}\left(\frac{\partial f_\alpha}{\partial\zeta} + \frac{\partial f}{\partial\zeta^\alpha}\right), \qquad \varepsilon_{00} = \frac{\partial f}{\partial\zeta} - 1$$

The corresponding linearized expressions for $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are found from (3.5)

$$A_{\alpha\beta} = \nabla_{(\alpha}u_{\beta)}, \quad B_{\alpha\beta} = \nabla_\alpha\nabla_\beta u_n, \quad u_\alpha = \frac{\partial r_0{}^i}{\partial\zeta^\alpha}u_i, \quad u_n = n_0{}^i u_i \qquad (8.3)$$

Let us take the hypothesis that the normal to the middle surface remains normal under deformation

$$f^\alpha = 0 \qquad (8.4)$$

and the function f has the form (*)(Footnote on the following page).

$$f = (1 + e)\zeta + {}^1\!/_2\,\chi\zeta^2 \qquad (8.5)$$

Then

$$\varepsilon_{\alpha\beta} = A_{\alpha\beta} - \zeta B_{\alpha\beta}$$

$$\varepsilon_{0\alpha} = \frac{1}{2}\zeta\left(\frac{\partial e}{\partial \xi^\alpha} + \frac{1}{2}\zeta\frac{\partial \chi}{\partial \xi^\alpha}\right), \qquad \varepsilon_{00} = e + \chi\zeta \tag{8.6}$$

Therefore, the parameter e has the meaning of deformation of a fiber normal to the middle surface, and χ is the gradient of fiber deformation.

We obtain for the averaged internal energy (the plate thickness is considered constant)

$$\begin{aligned}2\Phi = 2\int_{-h/2}^{h/2} \rho U\, d\zeta = h\{\lambda(A_\alpha^{\ \alpha})^2 + 2\mu A_{\alpha\beta}A^{\alpha\beta} - 2\lambda A_\alpha^{\ \alpha} e + (\lambda + 2\mu) e^2\} + \\ \frac{h^3}{12}\{\lambda(B_\alpha^{\ \alpha})^2 + 2\mu B_{\alpha\beta}B^{\alpha\beta} - 2\lambda B_\alpha^{\ \alpha}\chi + (\lambda + 2\mu)\chi^2\} + \\ \mu\frac{h^3}{12}a^{\alpha\beta}\frac{\partial e}{\partial \xi^\alpha}\frac{\partial e}{\partial \xi^\beta} + \mu\frac{h^5}{90}a^{\alpha\beta}\frac{\partial \chi}{\partial \xi^\alpha}\frac{\partial \chi}{\partial \xi^\beta}\end{aligned} \tag{8.7}$$

The last two terms can be substantial only on the edge of the plate, while they are of the order of h^2 / L^2 far from the edge as compared with $h(\lambda + 2\mu)e^2$ and $h^3(\lambda + 2\mu)\chi^2$ respectively, and they can be neglected (L is the distance whithin which e and χ vary by the characteristic value).

The tensile stress resultants and moments are defined by (5.5)

$$\begin{aligned}n^{\alpha\beta} = h(\lambda A_\sigma^{\ \sigma} a^{\alpha\beta} + 2\mu A^{\alpha\beta} - \lambda e a^{\alpha\beta}) \\ m^{\alpha\beta} = \frac{h^3}{12}(\lambda B_\sigma^{\ \sigma} a^{\alpha\beta} + 2\mu B^{\alpha\beta} - \lambda\chi a^{\alpha\beta})\end{aligned} \tag{8.8}$$

Equations(5.10),(5.11) become

$$\begin{aligned}\nabla_\alpha\nabla_\beta m^{\alpha\beta} = p_+ + p_- - \frac{h}{2}\nabla_\alpha(p_+^{\ \alpha} - p_-^{\ \alpha}) + \int_{-h/2}^{h/2}\rho F\, d\zeta + \nabla_\alpha\int_{-h/2}^{h/2}\rho F^\alpha\zeta\, d\zeta \\ \nabla_\beta n^{\alpha\beta} = -\left(p_+^{\ \alpha} + p_-^{\ \alpha} + \int_{-h/2}^{h/2}\rho F^\alpha\, d\zeta\right)\end{aligned} \tag{8.9}$$

The plus and minus subscripts here denote quantities on the surface $\zeta = h/2$ and $\zeta = -h/2$, respectively, p is the projection of the force acting on the surfaces $\zeta = \pm h/2$ in a direction normal to the plate, p^α is the projection of this same force on the middle plane of the plate, and the notation for the external mass forces F and F^α are analogous. The Euler equations (5.9) for the parameters e and χ reduce to

$$\begin{aligned}h[(\lambda + 2\mu)e - \lambda A_\alpha^{\ \alpha}] = \frac{h}{2}(p_+ - p_-) + \int_{-h/2}^{h/2}\rho F\zeta\, d\zeta \\ \frac{h^3}{12}[(\lambda + 2\mu)\chi - \lambda B_\alpha^{\ \alpha}] = \frac{h^2}{8}(p_+ + p_-) + \frac{1}{2}\int_{-h/2}^{h/2}\rho F\zeta^2\, d\zeta\end{aligned} \tag{8.10}$$

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*) It should be noted that the hypotheses resulting in the Kirchhoff model are formulated incorrectly in many monographs and papers. Namely, besides the condition (8.4), an assumption is made that the transverse fibers are not deformed. In reality, in the case of plate bending ($A_{\alpha\beta} = 0$, $e = 0$) deformation of a transverse fiber, described by the parameter χ yields a contribution of the same order of smallness to the elastic energy, as does deformation of the middle surface.

Eliminating the parameters e and χ from the system (8.8) – (8.10), we arrive at the equations of the theory of plates.

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BIBLIOGRAPHY

1. Sedov, L. I., Mathematical methods of constructing models of continuous media. Usp. Matem. Nauk, Vol. 20, №5, 1965.
2. Sedov, L. I., Models of continuous media with internal degrees of freedom. PMM Vol. 32, №5, 1968.
3. Berdichevskii, V. L. and Sedov, L. I., Dynamic theory of continuously distributed dislocations. Its relation to plasticity theory. PMM Vol. 31, №6, 1967.

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ON THE NORMALIZATION OF A HAMILTONIAN SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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We construct an algorithm for seeking a real canonic transformation of a linear Hamiltonian system of differential equations to normal form. As an example we consider the application of this transformation in the restricted three-body problem.

1. We consider the Hamiltonian system of differential equations

$$d\mathbf{x} / dt = \mathbf{I H}(t)\mathbf{x}, \qquad \mathbf{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) \tag{1.1}$$

The variables x_k and x_{n+k} are canonically conjugate (x_k are the coordinates, x_{n+k} are the momenta) in the corresponding mechanical problem. The $2n$ th-order symmetric matrix $\mathbf{H}(t)$ is assumed real, continuous, 2π-periodic in t. The matrix $\mathbf{I}$ has the form

$$\mathbf{I} = \begin{Vmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{Vmatrix}, \qquad (\mathbf{I}^{-1} = \mathbf{I}' = -\mathbf{I},\ \mathbf{I}^2 = -\mathbf{E},\ \det \mathbf{I} = 1)$$

where $\mathbf{E}$ is the n th-order unit matrix.

The solution of a linear system is usually chosen as the generating solution when investigating stability, analyzing nonlinear oscillations, constructing approximate solutions of nonlinear Hamiltonian systems. Therefore, it is desirable to choose those coordinates in which the solution of the linear system (1.1) is described most simply.

System (1.1), as also every linear system with continuous periodic coefficients, is reducible [1]. This means that there exists a linear change of variables with a continu-